

ON A THEOREM OF GRIGOR'YAN, HU AND LAU

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ABSTRACT. We refine a result of Grigor'yan, Hu and Lau to give a moment condition on a heat kernel which characterizes the critical exponent at which a family of Besov spaces associated to the Dirichlet energy becomes trivial.

Following [1], consider a metric measure space (M, d, μ) , with M nonempty and μ Borel, that admits a heat kernel $\{p_t\}_{t>0}$. The latter is a collection of symmetric, non-negative, measurable functions on $M \times M$ all of which have unit integral, satisfy the semigroup property $p_{t+s}(x, y) = \int p_s(x, z)p_t(z, y)d\mu(z)$ for all $s, t > 0$, and approximate the identity in the sense that if $f \in L^2$ then $\int p_t(x, y)f(y)d\mu(y) \rightarrow f$ in L^2 as $t \downarrow 0$. This hypothesis has many consequences, among which we will need that setting

$$T_t u(x) = \int_M p_t(x, y)u(y) d\mu(y), \text{ and}$$

$$\mathcal{E}_t(u) = t^{-1} \langle u - T_t u, u \rangle,$$

where \langle, \rangle is the L^2 inner product, we find $\mathcal{E}_t(u)$ is decreasing in $t > 0$ so $\mathcal{E}(u) = \lim_{t \downarrow 0} \mathcal{E}_t(u)$ exists, though it may be infinite. Moreover setting $\mathcal{D}(\mathcal{E}) = \{u \in L^2 : \mathcal{E}(u) < \infty\}$ we have that \mathcal{E} is a Dirichlet form with domain $\mathcal{D}(\mathcal{E})$.

In [1] the authors further assume the heat kernel has a two-sided estimate of the form

$$(1) \quad \frac{1}{t^{\alpha/\beta}} \Phi_1\left(\frac{d(x, y)}{t^{1/\beta}}\right) \leq p_t(x, y) \leq \frac{1}{t^{\alpha/\beta}} \Phi_2\left(\frac{d(x, y)}{t^{1/\beta}}\right)$$

for μ -a.e. $x, y \in M$ and all $t > 0$, where α and β are positive constants and Φ_1 and Φ_2 are non-negative monotone decreasing functions on $[0, \infty)$. They then show that α and β are determined by (M, d, μ) provided that $\Phi_1(1) > 0$ and Φ_2 satisfies a moment condition of the form

$$(H_\gamma) \quad \int_0^\infty s^\gamma \Phi_2(s) \frac{ds}{s} < \infty$$

for some suitable value of γ . Some of these results are stated in terms of a Besov space which they denote $W^{\sigma, 2}$ but which is sometimes called $\text{Lip}(\sigma, 2, \infty)$. To define this space let

$$W_\sigma(u) = \sup_{0 < r < 1} r^{-2\sigma} \int_M \int_{B(x, r)} |u(y) - u(x)|^2 d\mu(y) d\mu(x)$$

where $\int_B = \mu(B)^{-1} \int_B$ is the average and $B(x, r)$ is the ball of radius r with center x . Then let $W^{\sigma, 2} = \{u \in L^2 : W_\sigma(u) < \infty\}$. This is a Banach space with norm $\|u\|_2 + W_\sigma^{1/2}$. Also let $\beta^* = 2 \sup\{\sigma : \dim W^{\sigma, 2} = \infty\}$.

Among the main results in [1] are the following:

Theorem 1 ([1] Theorems 3.2, 4.2, 4.6).

Suppose (M, d, μ) has a heat kernel satisfying (1) and that $\Phi_1(1) > 0$.

- (1) If (H_γ) holds for $\gamma = \alpha$ then μ is Ahlfors regular with exponent α .

- (2) If (H_γ) holds for $\gamma = \alpha + \beta$ then $\mathcal{D}(\mathcal{E}) = W^{\beta/2,2}$ and $\mathcal{E}(u) \simeq W_{\beta/2}(u)$.
(3) If (H_γ) holds for $\gamma > \alpha + \beta$ then for $\sigma > \beta/2$ the space $W^{\sigma,2} = \{0\}$ and $\beta = \beta^*$.

The purpose of this note is to show that the third of the above implications may be improved as follows:

Theorem 2. Suppose (M, d, μ) has a heat kernel satisfying (1), that $\Phi_1(1) > 0$ and (H_γ) holds for $\gamma = \alpha + \beta$. Then $W^{\sigma,2} = \{0\}$ and $\beta = \beta^*$.

Proof. We follow the proof of Theorem 4.6 of [1]. They decompose $\mathcal{E}_t(u) = A(t) + B(t)$ where for $\epsilon = \sigma - \beta$

$$\begin{aligned}
B(t) &= \frac{1}{2t} \int_M \int_{B(x,1)} (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x) \\
&= \frac{1}{2t} \sum_{k=1}^{\infty} \int_M \int_{B(x, 2^{-(k-1)}) \setminus B(x, 2^{-k})} (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x) \\
&\leq \frac{t^{\epsilon/\beta}}{2} \sum_{k=1}^{\infty} \left(\frac{2^{-k}}{t^{1/\beta}} \right)^{\alpha+\beta+\epsilon} \Phi_2 \left(\frac{2^{-k}}{t^{1/\beta}} \right) 2^{k(\alpha+\sigma)} \int_M \int_{B(x, 2^{-(k-1)})} (u(x) - u(y))^2 d\mu(y) d\mu(x) \\
&\leq CW_\sigma(u) t^{\epsilon/\beta} \sum_{k=1}^{\infty} \left(\frac{2^{-k}}{t^{1/\beta}} \right)^{\alpha+\beta+\epsilon} \Phi_2 \left(\frac{2^{-k}}{t^{1/\beta}} \right) \\
&\leq CW_\sigma(u) t^{\epsilon/\beta} \int_0^{t^{-1/\beta}} s^{\alpha+\beta+\epsilon} \Phi_2(s) \frac{ds}{s}
\end{aligned}$$

in which the first inequality is from the upper bound in (1) and the second is from the definition of $W_\sigma(u)$ and the fact that part (1) of Theorem 1 implies $\mu(B(x, 2^{-(k-1)})) \simeq 2^{k\alpha}$.

The above is essentially shown in the proof of Theorem 4.6 in [1]; they then assume (H_γ) for $\gamma = \alpha + \beta + \epsilon$ to establish that the integral is bounded independent of t and conclude $\lim_{t \downarrow 0} B(t) = 0$. However this also follows from (H_γ) for $\gamma = \alpha + \beta$. This is actually a standard exercise: given $\delta > 0$ use (H_γ) for $\gamma = \alpha + \beta$ to obtain T so small that

$$\int_{T^{-1/\beta}}^{\infty} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s} < \delta$$

from which

$$\begin{aligned}
B(t) &\leq CW_\sigma(u) t^{\epsilon/\beta} \int_0^{T^{-1/\beta}} s^{\alpha+\beta+\epsilon} \Phi_2(s) \frac{ds}{s} + CW_\sigma(u) t^{\epsilon/\beta} \int_{T^{-1/\beta}}^{t^{-1/\beta}} s^{\alpha+\beta+\epsilon} \Phi_2(s) \frac{ds}{s} \\
&\leq CW_\sigma(u) \left(\frac{t}{T} \right)^{\epsilon/\beta} \int_0^{T^{-1/\beta}} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s} + CW_\sigma(u) \int_{T^{-1/\beta}}^{t^{-1/\beta}} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s} \\
&\leq CW_\sigma(u) \left(\frac{t}{T} \right)^{\epsilon/\beta} \int_0^{\infty} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s} + CW_\sigma(u) \delta
\end{aligned}$$

and $\lim_{t \downarrow 0} B(t)$ follows. Since it is established in equation (4.17) of [1] that $\lim_{t \downarrow 0} A(t) = 0$ we conclude

$$\mathcal{E}(u) = \lim_{t \downarrow 0} \mathcal{E}_t(u) = \lim_{t \downarrow 0} A(t) + B(t) = 0.$$

□

Remark. A similar argument is used for a slightly different purpose in [3], and a slightly less general result with the same proof is in [2]. Nonetheless this specific result does not seem to be known, and the weaker result in part (3) of Theorem 1 is frequently cited.

REFERENCES

- [1] Alexander Grigor'yan, Jiaxin Hu and Ka-Sing Lau *Heat kernels on metric measure spaces and an application to semilinear elliptic equations*. Trans. Amer. Math. Soc. **355** (2003), no. 5, 20652095.
- [2] Katarzyna Pietruska-Paľuba *On function spaces related to fractional diffusions on d -sets* Stoch. Stoch. Rep. **70**, 153164 (2000)
- [3] Katarzyna Pietruska-Paľuba *Heat kernels on metric spaces and a characterisation of constant functions* Man. Math. **115**, 389399 (2004)